Supplementary Material

In this supplementary section, we provide a proof for Lemma 1 from Sec. 3.1. As a reminder, the lemma is defined in terms of matrices $A \in \mathbb{R}^{3\times 3}$ which are related to the coefficient vectors as:

$$A = \begin{bmatrix} -2a_1 & -a_3 & -a_4 \\ -a_3 & -2a_2 & -a_5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

The statement of the lemma itself is reproduced below.

Lemma 1 : *Let A and $\widetilde{A}$ correspond to two coefficient matrices of the form in (6), and l and $\widetilde{l}$ to two lighting vectors. If,*

$$\frac{\bar{x}^T A^T l l^T A \bar{x}}{\bar{x}^T A^T A \bar{x}} = \frac{\bar{x}^T \widetilde{A}^T \widetilde{l}\widetilde{l}^T \widetilde{A} \bar{x}}{\bar{x}^T \widetilde{A}^T \widetilde{A} \bar{x}}, \forall \bar{x} \in \Omega, \tag{12}$$

Rank(V_Ω) = 15, Rank(A) $\geq$ 2, and $l^T A\bar{x} > 0, \forall \bar{x} \in \Omega$ (i.e., no point is in shadow), then

$$A^T l l^T A = \widetilde{A}^T \widetilde{l}\widetilde{l}^T \widetilde{A}, \quad A^T A = \widetilde{A}^T \widetilde{A}. \tag{13}$$

Moreover, if Rank(A) = 2, then Rank($\widetilde{A}$) = 2 and both A and $\widetilde{A}$ have a common null space.

The expression in (12) equates two rational forms in x. To prove the lemma, we will show that the equality holds for all x if we have a sufficient number of non-degenerate locations in the patch. Then, we will show that the corresponding coefficients in the quadratic expressions in the numerator and denominator must be equal when they are of the form in (6) and the conditions of Lemma 1 are met, essentially ruling out the possibility of a common factor or scaling term. To this end, we introduce another lemma, with proof, and then present the proof of Lemma 1.

Lemma 2. *Let $P, Q, \widetilde{P}, \widetilde{Q} \in \mathbb{R}^{3\times 3}$ be symmetric matrices. Then, $P = t\widetilde{P}$ and $Q = t\widetilde{Q}$, where $t \neq 0$ is a constant scalar, if*

$$\frac{\bar{x}^T P \bar{x}}{\bar{x}^T Q \bar{x}} = \frac{\bar{x}^T \widetilde{P} \bar{x}}{\bar{x}^T \widetilde{Q} \bar{x}}, \quad \forall \bar{x} \in \Omega, \tag{51}$$

Rank(V_Ω) = 15, and,

Case 1: *All of the following conditions are satisfied:*

$$q_{11} \neq 0, \quad q_{22} \neq 0, \quad q_{33} \neq 0, \tag{52}$$

$$4(p_{11}q_{12} - p_{12}q_{11})(p_{22}q_{12} - p_{12}q_{22}) + (p_{11}q_{22} - p_{22}q_{11})^2 \neq 0, \tag{53}$$

$$4(p_{11}q_{13} - p_{13}q_{11})(p_{33}q_{13} - p_{13}q_{33}) + (p_{11}q_{33} - p_{33}q_{11})^2 \neq 0, \tag{54}$$

$$4(p_{22}q_{23} - p_{23}q_{22})(p_{33}q_{23} - p_{23}q_{33}) + (p_{22}q_{33} - p_{33}q_{22})^2 \neq 0. \tag{55}$$

Case 2: *All of the following conditions are satisfied:*

$$p_{j2}, p_{2j}, q_{j2}, q_{2j}, \widetilde{p}_{j2}, \widetilde{p}_{2j}, \widetilde{q}_{j2}, \widetilde{q}_{2j} = 0, \quad \forall j \in \{1, 2, 3\}, \tag{56}$$

$$q_{11} \neq 0, \qquad q_{33} \neq 0, \tag{57}$$

$$4(p_{11}q_{13} - p_{13}q_{11})(p_{33}q_{13} - p_{13}q_{33}) + (p_{11}q_{33} - p_{33}q_{11})^2 \neq 0. \tag{58}$$

Proof of Lemma 2: We re-write (51) as

$$\left(\bar{x}_i^T P \bar{x}_i\right) \cdot \left(\bar{x}_i^T \widetilde{Q} \bar{x}_i\right) = \left(\bar{x}_i^T \widetilde{P} \bar{x}_i\right) \cdot \left(\bar{x}_i^T Q \bar{x}_i\right), \tag{59}$$

and note that this is fourth-order polynomial equation in $\bar{x}_i$. Combining these equations $\forall \bar{x}_i \in \Omega$, we have

$$V_\Omega C_{(P,Q,\widetilde{P},\widetilde{Q})} = 0, \tag{60}$$

where $C_{(P,Q,\widetilde{P},\widetilde{Q})} \in \mathbb{R}^{15}$ are the coefficients of the polynomial, and are of the form $(p_{ij}\widetilde{q}_{kl} - \widetilde{p}_{ij}q_{kl})$. Since V_Ω is rank 15, the above equation implies that $C_{(P,Q,\widetilde{P},\widetilde{Q})} = 0$. We now consider different sets of coefficients to prove the lemma.

Case 1. First look at the coefficients of $x^4, y^4, x^3y, xy^3, x^2y^2$:

$$x^4: \quad p_{11}\widetilde{q}_{11} = \widetilde{p}_{11}q_{11} \tag{61}$$

$$y^4: \quad p_{22}\widetilde{q}_{22} = \widetilde{p}_{22}q_{22} \tag{62}$$

$$x^3y: \quad p_{11}\widetilde{q}_{12} + p_{12}\widetilde{q}_{11} = \widetilde{p}_{11}q_{12} + \widetilde{p}_{12}q_{11} \tag{63}$$

$$xy^3: \quad p_{22}\widetilde{q}_{12} + p_{12}\widetilde{q}_{22} = \widetilde{p}_{22}q_{12} + \widetilde{p}_{12}q_{22} \tag{64}$$

$$x^2y^2: \quad p_{11}\widetilde{q}_{22} + 4p_{12}\widetilde{q}_{12} + p_{22}\widetilde{q}_{11} = \widetilde{p}_{11}q_{22} + 4\widetilde{p}_{12}q_{12} + \widetilde{p}_{22}q_{11} \tag{65}$$

Since $q_{11} \neq 0, q_{22} \neq 0$, we can define $t = \widetilde{q}_{11}/q_{11}$ and $s = \widetilde{q}_{22}/q_{22}$. Then (61) and (62) gives us

$$\widetilde{q}_{11} = q_{11}t, \quad \widetilde{p}_{11} = p_{11}t, \quad \widetilde{q}_{22} = q_{22}s, \quad \widetilde{p}_{22} = p_{22}s. \tag{66}$$

Substitute into (63), (64), and (65), we have

$$\begin{aligned} -q_{11}\widetilde{p}_{12} + p_{11}\widetilde{q}_{12} &= (p_{11}q_{12} - p_{12}q_{11})t, \\ (p_{12}q_{22} - p_{22}q_{12})s - q_{22}\widetilde{p}_{12} + p_{22}\widetilde{q}_{12} &= 0, \\ (p_{11}q_{22} - p_{22}q_{11})s - 4q_{12}\widetilde{p}_{12} + 4p_{12}\widetilde{q}_{12} &= (p_{11}q_{22} - p_{22}q_{11})t. \end{aligned} \tag{67}$$

This can be thought of as a linear system of equations on $(s, \widetilde{p}_{12}, \widetilde{q}_{12})$, with one obvious solution $(t, p_{12}t, q_{12}t)$. This solution will be unique when the corresponding coefficient matrix is non-singular, *i.e.*,

$$\det \begin{vmatrix} 0 & -q_{11} & p_{11} \\ (p_{12}q_{22} - p_{22}q_{12}) & -q_{22} & p_{22} \\ (p_{11}q_{22} - p_{22}q_{11}) & -4q_{12} & 4p_{12} \end{vmatrix} \neq 0. \tag{68}$$

Expanding this gives us (53), and therefore we have

$$[\widetilde{p}_{11}, \widetilde{p}_{12}, \widetilde{p}_{22}\widetilde{q}_{11}, \widetilde{q}_{12}, \widetilde{q}_{22}] = t[p_{11}, p_{12}, p_{22}, q_{11}, q_{12}, q_{22}]. \tag{69}$$

This approach can be used to show the same relationship for other terms in $P, Q, \widetilde{P}, \widetilde{Q}$. Specifically, the coefficients of $\{x^4, x^3, x^2, x, 1\}$ give us that $[q_{11}, q_{13}, q_{33}, p_{11}, p_{13}, p_{33}]$ and $[\widetilde{q}_{11}, \widetilde{q}_{13}, \widetilde{q}_{33}, \widetilde{p}_{11}, \widetilde{p}_{13}, \widetilde{p}_{33}]$ are proportional, and since they are linked by q_{11} and $\widetilde{q}_{11}$, the constant of proportionality must also be t. Similarly, looking at the coefficients of $\{y^4, y^3, y^2, y, 1\}$ gives us that $[q_{22}, q_{23}, q_{33}, p_{33}, p_{23}, p_{33}]$ and $[\widetilde{q}_{22}, \widetilde{q}_{23}, \widetilde{q}_{33}, \widetilde{p}_{33}, \widetilde{p}_{23}, \widetilde{p}_{33}]$ are proportional, with q_{22} and $\widetilde{q}_{22}$ linking the proportionality constant to t.

Case 2. For this case, we need to only look at the coefficients of $\{x^4, x^3, x^2, x, 1\}$, which gives us $[q_{11}, q_{13}, q_{33}, p_{11}, p_{13}, p_{33}] = t[\widetilde{q}_{11}, \widetilde{q}_{13}, \widetilde{q}_{33}, \widetilde{p}_{11}, \widetilde{p}_{13}, \widetilde{p}_{33}]$. ■

Proof of Lemma 1: Without loss of generality, we rotate and translate the co-ordinate system so that $a_3 = 0$, and $(0,0) \in \Omega$, and define $P = A^T l l^T A$, $Q = A^T A$, $\widetilde{P} = \widetilde{A}^T \widetilde{l}\,\widetilde{l}^T \widetilde{A}$ and $\widetilde{Q} = \widetilde{A}^T \widetilde{A}$. We consider two cases corresponding to the rank of A.

Case 1. Rank(A) = 3: We apply case 1 of Lemma 2 by showing that the conditions (52)-(55) hold:

1) Since A is invertible, we have $q_{11} = 4a_1^2 \neq 0$ and $q_{22} = 4a_2^2 \neq 0$. Also, $q_{33} = a_4^2 + a_5^2 + 1 \neq 0$, and therefore, (52) is satisfied.
2) For (53) to be satisfied, we need

$$256a_1^4 a_2^4 \left(l_x^2 + l_y^2\right)^2 \neq 0, \tag{70}$$

where $l = [l_x, l_y, l_z]$. Since A is invertible, $a_1 \neq 0$ and $a_2 \neq 0$. Note that (53) is violated if $l_x = l_y = 0$ and $\widetilde{l}_x = \widetilde{l}_y = 0$ (if not the latter, we can switch $\{a, l\}$, and $\{\widetilde{a}, \widetilde{l}\}$), but in that case, it is easy to see that

$$A^T ll^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_z^2 \end{bmatrix}, \ \widetilde{A}^T \widetilde{l}\, \widetilde{l}^T \widetilde{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{l}_z^2 \end{bmatrix}, \quad (71)$$

which in turn implies $A^T ll^T A = t\widetilde{A}^T \widetilde{l}\, \widetilde{l}^T \widetilde{A}$, with $t = l_z^2/\widetilde{l}_z^2$.

3) For (54)-(55) to be satisfied, we need

$$16a_1^4 \left((a_5^2+1)l_x^2 + (l_z - a_5 l_y)^2\right)^2 \neq 0, \quad (72)$$

$$16a_2^4 \left((a_4^2+1)l_y^2 + (l_z - a_4 l_x)^2\right)^2 \neq 0. \quad (73)$$

Since $a_1, a_2 \neq 0$, these conditions will be violated when $l_x = 0$ and $l_z - a_5 l_y = 0$; or $l_y = 0$ and $l_z - a_4 l_x = 0$, respectively. But these cases can be ruled out, since they result in the point $(0, 0)$ being in shadow.

Therefore, from case 1 of Lemma 2 we have that

$$A^T ll^T A = t\widetilde{A}^T \widetilde{l}\widetilde{l}^T \widetilde{A}, \quad A^T A = t\widetilde{A}^T \widetilde{A}. \quad (74)$$

To show $t = 1$, we first look at the top-left 2×2 block of the matrix

$$\begin{bmatrix} 4a_1^2 & \\ & 4a_2^2 \end{bmatrix} = t \begin{bmatrix} 4\widetilde{a}_1^2 + \widetilde{a}_3^2 & 2\widetilde{a}_3(\widetilde{a}_1 + \widetilde{a}_2) \\ 2\widetilde{a}_3(\widetilde{a}_1 + \widetilde{a}_2) & 4\widetilde{a}_2^2 + \widetilde{a}_3^2 \end{bmatrix}, \quad (75)$$

which implies

$$a_1 = p_1\sqrt{t}\widetilde{a}_1, \ a_2 = p_2\sqrt{t}\widetilde{a}_2, \ \widetilde{a}_3 = 0, \quad (76)$$

with $p_1 = \pm 1, p_2 = \pm 1$. Next compare the $(1, 3)$ and $(2, 3)$ entry of the matrices (74), we have

$$\begin{cases} 2a_1 a_4 = & 2t\widetilde{a}_1\widetilde{a}_4 \\ 2a_2 a_5 = & 2t\widetilde{a}_2\widetilde{a}_5 \end{cases}, \quad \Rightarrow \quad \begin{cases} a_4 = & p_1\sqrt{t}\widetilde{a}_4 \\ a_5 = & p_2\sqrt{t}\widetilde{a}_5 \end{cases}. \quad (77)$$

Finally, look at the $(3, 3)$ entry of the matrices in (74)

$$1 + a_4^2 + a_5^2 = t(1 + \widetilde{a}_4^2 + \widetilde{a}_5^2), \quad \Rightarrow \quad t = 1. \quad (78)$$

Case 2. Rank$(A) = 2$: Again, without loss of generality, we assume that the rank deficiency in A is caused by a_2 being equal to 0. Before we can apply case 2 of Lemma 2, we need to show that there is no possible solution for $(\widetilde{a}, \widetilde{l})$ where $\widetilde{a}_2 \neq 0$, or $\widetilde{a}_3 \neq 0$. To do so, we look at the expression for $I_{\bar{x}}$ in terms of a and l:

$$I_{\bar{x}} = \frac{-(2a_1 x + a_4)l_x - a_5 l_y + l_z}{\sqrt{(2a_1 x + a_4)^2 + a_5^2 + 1}}. \quad (79)$$

Note that the intensity here is independent of the coordinate y. Since $(\widetilde{a}, \widetilde{l})$ produce the same set of intensities, they too must be independent of y, which implies that $\widetilde{a}_2 = \widetilde{a}_3 = 0$.

We can then simply apply case 2 of Lemma 2, using the same approach as in case 1 above, where (56),(57) are directly satisfied by the constraints on a and that $\widetilde{a}_2, \widetilde{a}_3 = 0$, and (58) is satisfied by $a_1 \neq 0$, and the constraint that the point $(0, 0)$ not be in shadow. ∎