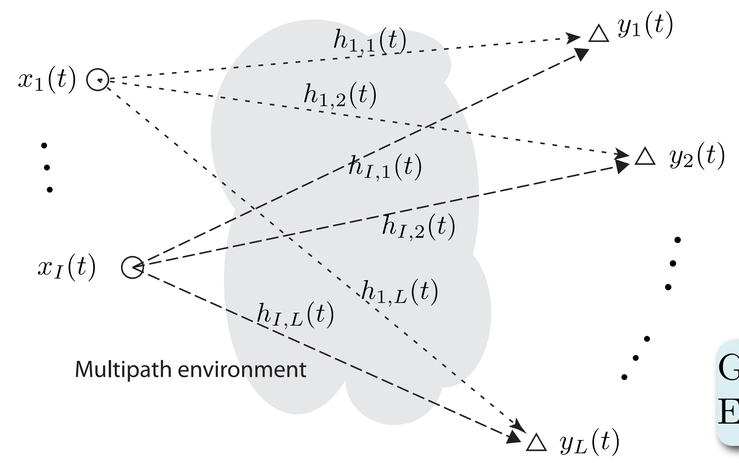
# Blind Estimation and Low-Rate Sampling of Sparse MIMO Systems with Common Support

Ying Xiong and Yue M. Lu

HARVARD School of Engineering and Applied Sciences

## Introduction

Multi-Input Multi-Output Systems



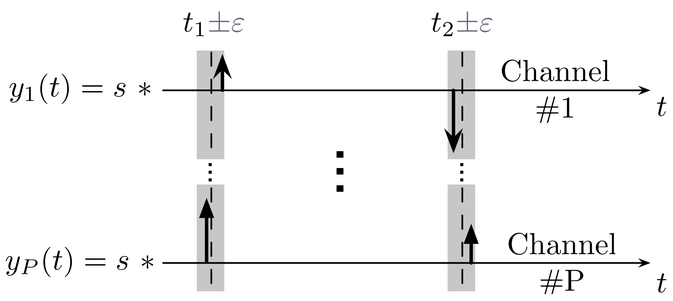

**Problem Statement**
In a multi-path MIMO system, given only the *sensor measurements*, how to jointly estimate the *input signals* and the *channel properties*?

Given: $\{y_\ell(t)\}$,
Estimate: $\{x_i(t), h_{i,\ell}(t)\}$.

**Assumptions**
Channels are *sparse* and have *common support*. This is a good approximation for most multi-path systems with limited bandwidth.

$$h_{i,\ell}(t) = \sum_{k=1}^{K} c_k^{(i,\ell)} \delta(t - t_k)$$

DFS




$$H_{i,\ell}[m] = \sum_{k=1}^{K} c_k^{(i,\ell)} u_k^m,$$

where $u_k \stackrel{\text{def}}{=} e^{-j2\pi t_k/T}$.

$$\Delta\Omega \cdot \Delta t \geq \frac{1}{2}$$

$$\Delta t_k \leq \frac{d}{c}$$

$$\Delta t_k \ll \Delta t \quad \Longleftarrow \quad \frac{d}{c} \ll \frac{1}{4\pi B}, \quad \text{i.e.} \quad d \cdot B \ll \frac{c}{4\pi} \approx 2.4 \times 10^7 \text{ (m/s)}$$

## Preliminaries

### MIMO System Modeling

Time Domain Convolution

$$y_\ell(t) = \sum_{i=1}^{I} h_{i,\ell}(t) * x_i(t),$$

DFS

Frequency Domain Formulation

$$\boldsymbol{Y}[m] = \boldsymbol{H}[m]\,\boldsymbol{X}[m]$$

### Inherent Ambiguities

The MIMO systems have inherent ambiguities:

- A *time shift ambiguity* such that if we shift input signals to some constant time $\tau$ and shift the channel response with $-\tau$, the shift can not be observed in the output signals.
- A *scale ambiguity* such that if we rearrange the input signals with matrix $\boldsymbol{E}$, and rearrange the channels with $\boldsymbol{E}^{-1}$, the rearrangement cannot be observed in the output signals.

$$\{\boldsymbol{X}[m], \boldsymbol{H}[m]\} \xrightarrow{\text{Ambiguity}} \{\xi^{-m}\boldsymbol{E}^{-1}\boldsymbol{X}[m], \xi^{m}\boldsymbol{H}[m]\boldsymbol{E}\}$$

## Blind Estimation

### Considering a SIMO case

**Proposition 1** *In a SIMO system with SCS channels, if the number of sensors $L$ is greater than or equal to the cardinality of the channel support $K$, i.e.,*

$$L \geq K$$

*and if there exists a subband of at least $K+3$ continuous Fourier coefficients such that $X[m] \neq 0$ for $m_0 \leq m < m_0 + K + 3$, then the system can be fully resolved up to two free parameters, namely an amplitude ambiguity $e$ and a delay ambiguity $\tau$.*

### Sketch of Proof

$$H_\ell[m] = Y_\ell[m]/X[m].$$

$$\sum_{k=0}^{K} a_k \boldsymbol{H}[m-k] = \boldsymbol{0}.$$

Generalized Annihilating Filter

Combining input in all channels

$$\sum_{k=0}^{K} a_k \frac{Y_\ell[m-k]}{X[m-k]} = \sum_{k=0}^{K} Y_\ell[m-k] b_{k,m} = 0.$$

Known $Y_\ell[m]$, solve for $b_{k,m} = \frac{a_k}{X[m-k]}$

$$\widetilde{\boldsymbol{B}} = (\widetilde{b}_{k,m}) = \begin{bmatrix} \frac{a_0 d_m}{X[m]} & \frac{a_0 d_{m+1}}{X[m+1]} & \frac{a_0 d_{m+2}}{X[m+2]} & \cdots \\ \frac{a_1 d_m}{X[m-1]} & \frac{a_1 d_{m+1}}{X[m]} & \frac{a_1 d_{m+2}}{X[m+1]} & \cdots \\ \frac{a_2 d_m}{X[m-2]} & \frac{a_2 d_{m+1}}{X[m-1]} & \frac{a_2 d_{m+2}}{X[m]} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \boldsymbol{\Lambda}_a \begin{bmatrix} \frac{1}{X[m]} & \frac{1}{X[m+1]} & \frac{1}{X[m+2]} & \cdots \\ \frac{1}{X[m-1]} & \frac{1}{X[m]} & \frac{1}{X[m+1]} & \cdots \\ \frac{1}{X[m-2]} & \frac{1}{X[m-1]} & \frac{1}{X[m]} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \boldsymbol{\Lambda}_d,$$

$$\frac{\widetilde{b}_{k-1,m}\widetilde{b}_{k+1,m+1}}{\widetilde{b}_{k,m}\widetilde{b}_{k,m+1}} = \frac{a_{k-1}a_{k+1}}{a_k^2} \stackrel{\text{def}}{=} s_k,$$

$$a_k = \left(\prod_{j=1}^{k-1} s_j^{k-j}\right) a_1^k.$$

Factorizing matrix to compute annihilating filter coefficients, then the system can be linearly solved.

### Extension to MIMO case

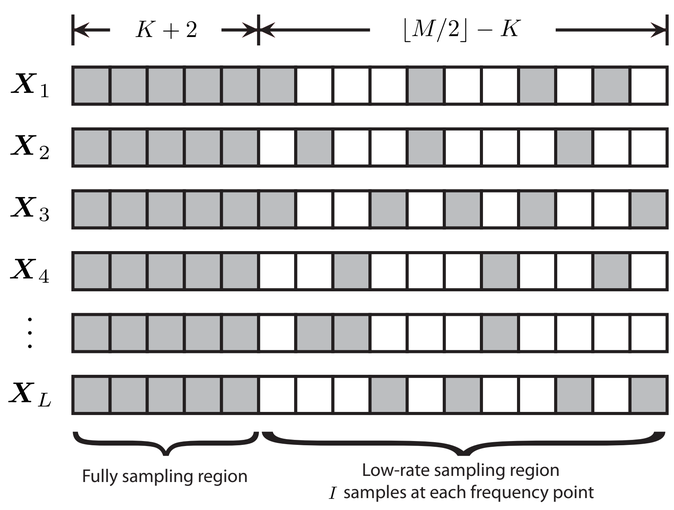
**Proposition 2** *In a MIMO system with SCS channels, let $I$ be the number of sources, $L$ the number of sensors and $K$ the cardinality of the channel support. If*

$$L \geq KI,$$

*and if there exists a subband of at least $K+3I$ frequency indices such that (15) holds for $m_0 + 1 \leq m < m_0 + K + 3I$, then the MIMO system can be fully resolved up to an amplitude ambiguity matrix $\boldsymbol{E}$ and a delay ambiguity $\tau$.*

## Low-Rate Sampling

**Proposition 3** *Under the same condition as stated in Proposition 2, perfect reconstruction on all the sensor measurements can be achieved with probability one, given that we keep $L \geq KI$ sensor samples on a subband of $K+3I$ frequency indices and $L' \geq I$ sensor samples on all the other frequency indices.*



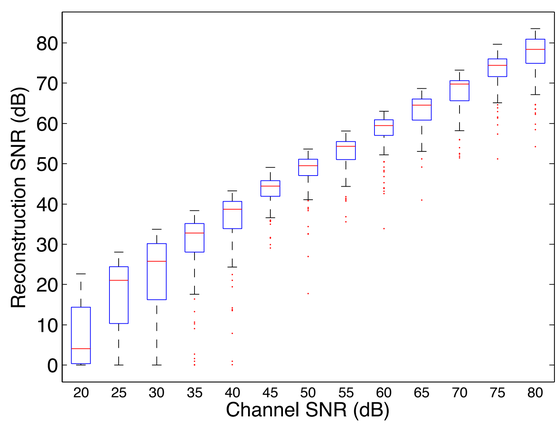

Observations: the multiple channels in the output are highly correlated.

Efficient Low-rate Sampling:

- Fully sampling in a small sub-band of the output signals and use them to recover all the channel properties
- Only a sparse sensing on the rest of frequency band is necessary to fully recover the input signals.

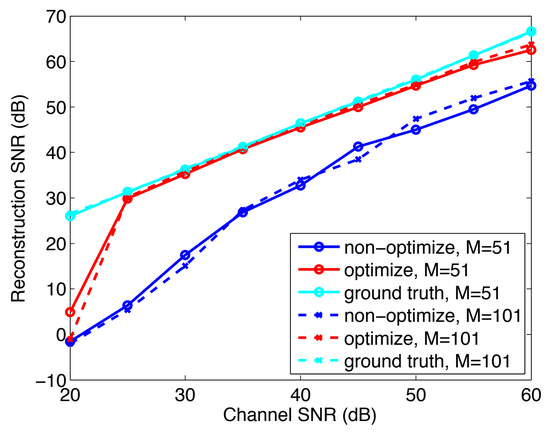
## Experimental Results

Input-Output SNR Graph




Reconstruction result of a MIMO system. Experiments are performed 100 times at each SNR point, and the box plots show the median (red lines), 25th and 75th percentile (blue boxes) and the extreme data points (black lines).

Applying Nonlinear Optimization




Reconstruction result of a SIMO system by three different approaches. The first (blue, bottom) uses estimated time directly; the second (red, middle) uses optimized time by a nonlinear minimization; and the last (cyan, top) uses the ground truth time.

## Summary

- A novel algorithm solving MIMO systems with *sparse common support*.
- *Blindly estimate* the unknown source signals and the channel information by *using only the sensor measurements*.
- Derived an efficient *low-rate* sampling scheme that significantly reduces the number of samples.