Spherical Harmonics

Ying Xiong
School of Engineering and Applied Sciences
Harvard University
yxiong@seas.harvard.edu

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1 Definitions

Note that different conventions on this topic exist. In this document, we follow the one in Wikipedia (at the time this document is created), which is also consistent with that of Matlab (R2013b).

1.1 Associated Legendre polynomials

The associated Legendre polynomials are defined as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} \left(1-x^2\right)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} \left(x^2-1\right)^\ell, \quad \text{for } -1 \le x \le 1,\ \ell \ge 0,\ -\ell \le m \le \ell. \tag{1}$$

This definition has the property that

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x). \tag{2}$$

Matlab also provides implementation for "fully normalized associated Legendre functions", defined as

$$N_\ell^m = (-1)^m \sqrt{\frac{\left(\ell+\frac{1}{2}\right)!\,(\ell-m)!}{(\ell+m)!}} P_\ell^m \tag{3}$$

1.1.1 Low-order examples

Associated Legendre polynomials with $\ell \le 2$.

$$\begin{array}{ccccc} & & P_0^0(x) = 1, & & \\ & P_1^{-1}(x) = -\frac{1}{2}P_1^1(x), & P_1^0(x) = x, & P_1^1(x) = -\left(1-x^2\right)^{1/2}, & \\ P_2^{-2}(x) = \frac{1}{24}P_2^2(x), & P_2^{-1}(x) = -\frac{1}{6}P_2^1(x), & P_2^0(x) = \frac{1}{2}(3x^2-1), & P_2^1(x) = -3x(1-x^2)^{1/2}, & P_2^2(x) = 3(1-x^2). \end{array}$$

1.2 Spherical harmonic functions

The spherical harmonic functions are defined as

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)\, e^{im\phi}, \tag{4}$$

or equivalently by fully normalized associate Legendre functions

$$Y_\ell^m\left(\theta,\phi\right) = \left(-1\right)^m \sqrt{\frac{1}{2\pi}} N_\ell^m\left(\cos\theta\right) e^{im\phi}. \tag{5}$$

A real basis of spherical harmonic functions can be defined as below

$$Y_{\ell,m} = \begin{cases} \frac{1}{\sqrt{2}}\left(Y_\ell^{-m} + \left(-1\right)^m Y_\ell^m\right) & \text{if } m > 0 \\ Y_\ell^0 & \text{if } m = 0 \\ \frac{i}{\sqrt{2}}\left(Y_\ell^m - \left(-1\right)^m Y_\ell^{-m}\right) & \text{if } m < 0. \end{cases} \tag{6}$$

Note we use two subscripts for real basis, as opposed to one subscript and one super-script for complex one. The real basis can usually be more conveniently expressed in terms of $(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

1.2.1 Low-order examples

Spherical harmonic functions with $\ell \leq 2$.

$$\begin{array}{ccccc} & & Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}, & & \\ & Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}, & Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, & Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi}, & \\ Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi}, & Y_2^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi}, & Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right), & Y_2^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi}, & Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}. \end{array}$$

Real spherical harmonic functions with $\ell \leq 2$.

$$\begin{array}{ccccc} & & Y_{0,0} = \frac{1}{2}\sqrt{\frac{1}{\pi}}, & & \\ & Y_{1,-1} = \frac{1}{2}\sqrt{\frac{3}{\pi}}y, & Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}z, & Y_{1,1} = \frac{1}{2}\sqrt{\frac{3}{\pi}}x, & \\ Y_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xy, & Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}yz, & Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3z^2 - 1\right), & Y_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xz, & Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{\pi}}\left(x^2 - y^2\right). \end{array}$$

1.3 Inner product in spherical space

The inner product in the spherical space is defined as

$$\langle f, g\rangle = \int_{\mathbb{S}^2} f(\omega)\bar{g}(\omega)d\omega = \int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi} f(\theta,\phi)\bar{g}(\theta,\phi)\sin\theta\, d\phi\, d\theta. \tag{7}$$

By our convention, $\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = \delta_{\ell,\ell'}\delta_{m,m'}$.

1.4 Decomposition of a spherical function

Any well-behaved spherical function f can be decomposed as

$$f = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}, \tag{8}$$

where

$$f_\ell^m = \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\omega)\bar{Y}_\ell^m(\omega)d\omega, \tag{9}$$

$$f_{\ell,m} = \langle f, Y_{\ell,m}\rangle = \int_{\mathbb{S}^2} f(\omega)Y_{\ell,m}(\omega)d\omega. \tag{10}$$

2 Applications in Lambertian Reflectance [1, 2, 3]

We use the real form spherical harmonics in this application.

2.1 Convolution and Funke-Hecke theorem [1]

Let $L(\omega)$ be a spherical function and $k(x)$ a scalar function, we define "convolution" as

$$E(\omega_1) = (k * L)(\omega_1) \overset{\text{def}}{=} \int_{\mathbb{S}^2} k(\omega_1 \cdot \omega_2) L(\omega_2) d\omega_2, \tag{11}$$

where the result $E(\omega_1)$ is another spherical function.

Theorem 1 (Funk-Hecke)**.** *Let $k(x)$ be a bounded integrable function on $[-1, 1]$, whose Harmonic expansion*

$$k(\cos\theta) = \sum_{\ell=0}^{\infty} k_\ell Y_{\ell,0}(\theta, \phi) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} k_\ell P_\ell^0(\cos\theta). \tag{12}$$

Then we have

$$k * Y_{\ell,m} = \sqrt{\frac{4\pi}{2\ell+1}} k_l Y_{\ell,m}. \tag{13}$$

The theorem implies that

$$E = k * L = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\sqrt{\frac{4\pi}{2\ell+1}} k_\ell L_{\ell,m} \right) Y_{\ell,m}. \tag{14}$$

2.2 Lambertian Reflectance [2]

Let L denote distant lighting distribution, and $E(\mathbf{n})$ the irradiance of the surface normal $\mathbf{n}$. Then we have

$$E(\mathbf{n}) = \int_{\Omega(\mathbf{n})} L(\omega) (\mathbf{n} \cdot \omega) d\omega \tag{15}$$

where $\mathbf{n}$ and ω are unit direction vectors and $\Omega(\mathbf{n})$ is the upper hemisphere of $\mathbf{n}$. Define

$$A(\theta) = \max[\cos\theta, 0] = \sum_{\ell=0}^{\infty} A_\ell Y_{\ell,0}(\theta, \phi). \tag{16}$$

Then we have

$$E(\mathbf{n}) = \int_{\mathbb{S}^2} L(\omega) A(\mathbf{n} \cdot \omega) d\omega = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} A_\ell L_{\ell,m} Y_{\ell.m}(\mathbf{n}), \tag{17}$$

or equivalently,

$$E_{\ell,m} = \sqrt{\frac{4\pi}{2\ell+1}} A_\ell L_{\ell,m} = \widehat{A}_\ell L_{\ell,m}, \quad \text{with } \widehat{A}_\ell = \sqrt{\frac{4\pi}{2\ell+1}} A_\ell. \tag{18}$$

The observation is that $\widehat{A}_\ell$ decays fast with respect to ℓ, and therefore the first several spherical harmonics (*e.g.* $\ell \leq 2$) captures most of the energy in $E(\mathbf{n})$.

The lighting coefficients can be found by an integration (pre-filtering)

$$L_{\ell,m} = \int_{\theta=0}^{\pi} \int_{\phi=-\pi}^{\pi} L(\theta, \phi) Y_{\ell,m}(\theta, \phi) \, \sin\theta \, d\theta \, d\phi. \tag{19}$$

2.3 Some Heuristics

2.3.1 $A(\theta)$ as a low-pass filter

The irradiance E is the convolution of lighting L and the function $A(\theta)$, which serves as a low-pass filter (because $\widehat{A}_\ell$ decays fast with respect to ℓ).

If the $A(\theta)$ were simply $\cos\theta$, *i.e.* without the zero clipping, it will be a perfect low-pass filter, because only the first order component would be non-zero.

2.3.2 Distant point light source

Now we consider a special case when lighting L is a distant point light source, which is equivalent to a directional light source. It can be written as a delta function

$$L(\theta, \phi) = \delta_{\theta_0,\phi_0}(\theta, \phi). \tag{20}$$

If the function $A(\theta)$ were simply $\cos\theta$, the rendering function is exactly captured by the first-order spherical harmonics, because $\hat{A}_\ell = 0$ for $\ell = 0$ and $\ell \geq 2$. In this case, the delta function will produce the same response $E(\mathbf{n})$ as the corresponding combination of first-order spherical harmonics.

In reality, when $A(\theta)$ has zero-clipping, the higher-order component will not be entirely cut-off and will need to be taken into consideration. The same first-order spherical harmonics will only produce exact $E\left(\mathbf{n}\right)$ on the hemispher $\Omega\left(\mathbf{n}\right)$, but not the whole sphere $\mathbb{S}^2$; and the best decomposition in the whole sphere can be different from that of the hemisphere.

References

[1] Ronen Basri and David W Jacobs. Lambertian reflectance and linear subspaces. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 25(2):218–233, 2003.

[2] Ravi Ramamoorthi and Pat Hanrahan. An efficient representation for irradiance environment maps. In *Proceedings of the 28th annual conference on Computer graphics and interactive techniques*, pages 497–500. ACM, 2001.

[3] Ravi Ramamoorthi and Pat Hanrahan. On the relationship between radiance and irradiance: determining the illumination from images of a convex lambertian object. *JOSA A*, 18(10):2448–2459, 2001.