Quaternion and Spatial Rotation

Ying Xiong
School of Engineering and Applied Sciences
Harvard University
yxiong@seas.harvard.edu

Created: February 9th, 2014.

Most of this notes come from Wikipedia [1, 2].

1 Definition

A quaternion q is represented as a 4-tuple (a, b, c, d), with basis $\{1, i, j, k\}$, written as

$$\mathbf{q} = (a, b, c, d) = a + b\,i + c\,j + d\,k. \tag{1}$$

The basis elements have multiplication property

$$i^2 = j^2 = k^2 = ijk = -1, \tag{2}$$

$$ij = k, \quad jk = i, \quad ki = j, \tag{3}$$

$$ji = -k, \quad kj = -\,i, \quad ik = -j. \tag{4}$$

The *Hamilton product* of two general quaternion is

$$\begin{aligned}&(a_1, b_1, c_1, d_1)\,(a_2, b_2, c_2, d_2)\\ =(\ &a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2\ ,\\ &a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2\ ,\\ &a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2\ ,\\ &a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2\ ).\end{aligned} \tag{5}$$

A quaternion can be divided into a *scalar part* and a *vector part*

$$\mathbf{q} = (r, \mathbf{v}), \quad \text{with } r \in \mathbb{R},\ \mathbf{v} \in \mathbb{R}^3. \tag{6}$$

We also consider scalar r and 3-vector $\mathbf{v}$ special forms of quaternion

$$\mathbf{q}_r = (r, \mathbf{0}), \quad \mathbf{q}_\mathbf{v} = (0, \mathbf{v}), \tag{7}$$

and write $\mathbf{q}_r$ and r ($\mathbf{q}_\mathbf{v}$ and $\mathbf{v}$) interchangably in this note.

For quaternion $\mathbf{q}$ defined in (1), its *conjugate* is

$$\mathbf{q}^* = a - b\,i - c\,j - d\,k, \tag{8}$$

its *norm* is

$$\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{\mathbf{q}^*\mathbf{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}, \tag{9}$$

and its *reciprocal* is

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2}, \quad \mathbf{q}\mathbf{q}^{-1} = \mathbf{q}^{-1}\mathbf{q} = 1. \tag{10}$$

Note that the multiplications in (9) and (10) are Hamilton product.

2 Spatial Rotation

Given a unit vector $\widehat{\mathbf{u}} = (u_x, u_y, u_z)$ with a scalar angle θ, we define quaternion

$$\mathbf{q} = e^{\frac{1}{2}\theta(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\right), \tag{11}$$

then for any given vector $\mathbf{p}$, its rotation across $\widehat{\mathbf{u}}$ axis for angle θ is

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}, \tag{12}$$

using Hamilton product. Note that both $\mathbf{q}$ and $-\mathbf{q}$ performs the same rotation.

2.1 Conversion to rotation matrix

Given a unit quaternion $\mathbf{q} = (a, b, c, d)$, it can be converted to a rotation matrix as

$$\boldsymbol{R} = \begin{bmatrix} 1-2c^2-2d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & 1-2b^2-2d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & 1-2b^2-2c^2 \end{bmatrix}. \tag{13}$$

To convert from a rotation matrix $\boldsymbol{R}$ to a quaternion,

$$\begin{aligned} \mathbf{q} =(& \frac{1}{2}\sqrt{R_{11}+R_{22}+R_{33}+1}, \\ = & \frac{1}{2}\sqrt{R_{11}-R_{22}-R_{33}+1}\,\text{sign}(R_{32}-R_{23}), \\ = & \frac{1}{2}\sqrt{-R_{11}+R_{22}-R_{33}+1}\,\text{sign}(R_{13}-R_{31}), \\ = & \frac{1}{2}\sqrt{-R_{11}-R_{22}+R_{33}+1}\,\text{sign}(R_{21}-R_{12})\ ). \end{aligned} \tag{14}$$

This conversion can also be implemented with a single square root, but one need to take special care on numerical stability when doing so.

References

[1] Wikipedia, "Plagiarism — Wikipedia, the free encyclopedia," 2014, [Online; accessed 9-February-2014]. [Online]. Available: https://en.wikipedia.org/wiki/Quaternion

[2] ——, "Plagiarism — Wikipedia, the free encyclopedia," 2014, [Online; accessed 9-February-2014]. [Online]. Available: https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation